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THE FORMS OF THE RELATION BETWEEN THE STRESS AND STRAIN TENSORS IN A NON-LINEARLY ELASTIC MATERIAL†

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New relations for the stress and strain tensors, which comprise energy pairs, are obtained for a non-linearly elastic material using a similar method to that employed by Novozhilov, based on a trigonometric representation of the tensors. Shear strain and stress tensors, not used previously, are introduced in a natural way. It is established that the unit tensor, the deviator and the shear tensor form an orthogonal tensor basis. The stress tensor can be expanded in a strain-tensor basis and vice versa. By using this expansion, the non-linear law of elasticity can be written in a compact and physically clear form. It is shown that in the frame of the principal axes the stresses are expressed in terms of the strains and vice versa using linear relations, while the non-linearity is contained in the coefficients, which are functions of mixed invariants of the tensors, introduced by Novozhilov, the generalized moduli of bulk compression and shear and the phase of similitude of the deviators. Relations for different energy pairs of tensors are considered, including for tensors of the true stresses and strains, where the generalized moduli of elasticity have a physical meaning for large strains.

1. The non-linear law of elasticity is usually constructed for Green's strain tensor $\mathbf{E}$ and the Piola–Kirchhoff stress tensor Σ. These tensors are symmetrical and coaxial and form an energy pair; the increment in the work of deformation can be written in the form $\delta W = \mathbf{\Sigma} : \delta\mathrm{E}$ (the two dots denote the operation of convolution).

We distinguish in the tensors a spherical and deviatoric part

$$\mathbf{B} = \frac{1}{3}e\mathbf{I} + \mathbf{D}_E, \quad \mathbf{\Sigma} = \frac{1}{3}\sigma\mathbf{I} + \mathbf{D}_\Sigma \tag{1.1}$$

where $\mathbf{I}$ is the unit tensor and $e = \operatorname{tr}\mathbf{E}$, $\sigma = \operatorname{tr}\mathbf{\Sigma}$ are the first invariants of the tensors. In addition to the deviators of the tensors $\mathbf{D}_E$ and $\mathbf{D}_\Sigma$ we will introduce two new tensors $\mathbf{S}_E$ and $\mathbf{S}_\Sigma$, which have not been needed previously. In the frame of principal the deformation axes we have

$$\begin{aligned}
\mathbf{D}_E &= \left(\varepsilon_i - \frac{1}{3}e\right)\mathbf{e}_i\mathbf{e}_i, \quad \mathbf{S}_E = \frac{1}{\sqrt{3}}(\varepsilon_k - \varepsilon_j)\mathbf{e}_i\mathbf{e}_i. \\
\mathbf{D}_\Sigma &= \left(\sigma_i - \frac{1}{3}\sigma\right)\mathbf{e}_i\mathbf{e}_i, \quad \mathbf{S}_\Sigma = \frac{1}{\sqrt{3}}(\sigma_k - \sigma_j)\mathbf{e}_i\mathbf{e}_i
\end{aligned} \tag{1.2}$$

where $\mathbf{e}_i$ are vectors of the principal orthonormalised basis, the subscripts $i, j, k = 1, 2, 3$ form a cyclic permutation and summation is carried out over the subscript i.

The components of the tensors $\mathbf{S}_E$ and $\mathbf{S}_\Sigma$ are proportional to the principal shear deformations and the principal shear stresses, and hence we will call them shear tensors for the strains and stresses, respectively. The following relations are satisfied for tensors (1.2)

$$\begin{aligned}
&\mathbf{D}_E : \mathbf{I} = \mathbf{S}_E : \mathbf{I} = 0, \quad \mathbf{D}_\Sigma : \mathbf{I} = \mathbf{S}_\Sigma : \mathbf{I} = 0 \\
&\mathbf{D}_E : \mathbf{S}_E = 0, \quad \mathbf{D}_E : \mathbf{D}_E = \mathbf{S}_E : \mathbf{S}_E = 2\bar{e}^2 \\
&\mathbf{D}_\Sigma : \mathbf{S}_\Sigma = 0, \quad \mathbf{D}_\Sigma : \mathbf{D}_\Sigma = \mathbf{S}_\Sigma : \mathbf{S}_\Sigma = 2\bar{\sigma}^2 \\
&\mathbf{D}_E^2 + \mathbf{S}_E^2 = \frac{4}{3}\bar{e}^2\mathbf{I}, \quad \mathbf{D}_\Sigma^2 + \mathbf{S}_\Sigma^2 = \frac{4}{3}\bar{\sigma}^2\mathbf{I}
\end{aligned} \tag{1.3}$$

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where $\bar{e}$ and $\bar{\sigma}$ are the intensities of the shear strains and shear stresses.

It follows from these relations that the tensors ($\mathbf{I}$, $\mathbf{D}_E$, $\mathbf{S}_E$) and ($\mathbf{I}$, $\mathbf{D}_\Sigma$, $\mathbf{S}_\Sigma$) form orthogonal tensor bases.

Formulae (1.3) are invariant with respect to the system of coordinates, and hence the last two equations of (1.3) can be regarded as the definition of the shear tensors in terms of the deviators in a system of coordinates of general form.

We note some of the properties of the deviators and shear tensors, expressed by (1.3): the convolution of the deviator and the shear tensor is equal to zero, these tensors are equal in norm (by the norm we mean convolution), and the sum of the squares of the deviator and the shear tensor is proportional to the unit tensor.

We will write the principal values of the deviators and shear tensors in trigonometric form

$$\varepsilon_i - \frac{1}{3}e = \frac{2}{\sqrt{3}}\bar{e}\cos\varphi_i, \quad \frac{1}{\sqrt{3}}(\varepsilon_k - \varepsilon_j) = -\frac{2}{\sqrt{3}}\bar{e}\sin\varphi_i$$
$$\sigma_i - \frac{1}{3}\sigma = \frac{2}{\sqrt{3}}\bar{\sigma}\cos\psi_i, \quad \frac{1}{\sqrt{3}}(\sigma_k - \sigma_j) = -\frac{2}{\sqrt{3}}\bar{\sigma}\sin\psi_i \qquad (1.4)$$

where the subscripts $i, j, k = 1, 2, 3$ form a cyclic permutation. The angles φ_i and ψ_i for different values of $i = 1, 2, 3$ differ by constant numbers that are multiples of $2\pi/3$. We will also further assume that the angles φ_i and ψ_i are read in the deviator plane $e = 0$ and $\sigma = 0$ from the origins that are chosen in the same way (this is possible since the tensors $\mathbf{E}$ and $\boldsymbol{\Sigma}$ are coaxial), in which case $\omega^* = \psi_i - \varphi_i$. The angle ω^* is called the phase of similitude of the tensor deviators. The formulae can be represented in tensor form

$$\mathbf{D}_E = \frac{2}{\sqrt{3}}\bar{e}\cos\mathbf{A}_E, \quad \mathbf{S}_E = -\frac{2}{\sqrt{3}}\bar{e}\sin\mathbf{A}_E$$
$$\mathbf{D}_\Sigma = \frac{2}{\sqrt{3}}\bar{\sigma}\cos\mathbf{A}_\Sigma, \quad \mathbf{S}_\Sigma = -\frac{2}{\sqrt{3}}\bar{\sigma}\sin\mathbf{A}_\Sigma$$

Here we have introduced tensors of the angles of the form $\mathbf{A}_E$ and $\mathbf{A}_\Sigma$, which have components φ_i and ψ_i in the frame of the principal axes. The difference of these tensors is proportional to the unit tensor: $\mathbf{A}_\Sigma - \mathbf{A}_E = \omega^*\mathbf{I}$.

In the common deviator plane of the tensors considered, we obtain the stress and strain vectors ($\mathbf{n}$ is the unit vector of the normal to this plane)

$$\mathbf{E}\cdot\mathbf{n} = \frac{1}{3}e\mathbf{n} + \sqrt{\frac{2}{3}}\bar{e}\boldsymbol{\nu}_e, \quad \mathbf{D}_E\cdot\mathbf{n} = \sqrt{\frac{2}{3}}\bar{e}\boldsymbol{\nu}_e, \quad \mathbf{S}_E\cdot\mathbf{n} = \sqrt{\frac{2}{3}}\bar{e}\mathbf{t}_e$$
$$\mathbf{n}\cdot\boldsymbol{\Sigma} = \frac{1}{3}\sigma\mathbf{n} + \sqrt{\frac{2}{3}}\bar{\sigma}\boldsymbol{\nu}_\sigma, \quad \mathbf{n}\cdot\mathbf{D}_\Sigma = \sqrt{\frac{2}{3}}\bar{\sigma}\boldsymbol{\nu}_\sigma, \quad \mathbf{n}\cdot\mathbf{S}_\Sigma = \sqrt{\frac{2}{3}}\bar{\sigma}\mathbf{t}_\sigma$$

It can be shown that the vectors ($\mathbf{n}$, $\boldsymbol{\nu}_e$, $\mathbf{t}_e$) and ($\mathbf{n}$, $\boldsymbol{\nu}_\sigma$, $\mathbf{t}_\sigma$) form orthogonal bases and ω^* is the angle between the vectors $\boldsymbol{\nu}_e$, $\boldsymbol{\nu}_\sigma$.

The deviator and the shear tensor for the strain tensor in the vector basis of the deviator plane have the form

$$\mathbf{D}_E = \frac{3D_e}{2\bar{e}^2}(\boldsymbol{\nu}_e\boldsymbol{\nu}_e - \mathbf{t}_e\mathbf{t}_e) - \frac{3S_e}{2\bar{e}^2}(\boldsymbol{\nu}_e\mathbf{t}_e + \mathbf{t}_e\boldsymbol{\nu}_e) + \sqrt{\frac{2}{3}}\bar{e}(\boldsymbol{\nu}_e\mathbf{n} + \mathbf{n}\boldsymbol{\nu}_e)$$
$$\mathbf{S}_E = -\frac{3S_e}{2\bar{e}^2}(\boldsymbol{\nu}_e\boldsymbol{\nu}_e - \mathbf{t}_e\mathbf{t}_e) - \frac{3D_e}{2\bar{e}^2}(\boldsymbol{\nu}_e\mathbf{t}_e + \mathbf{t}_e\boldsymbol{\nu}_e) + \sqrt{\frac{2}{3}}\bar{e}(\mathbf{t}_e\mathbf{n} + \mathbf{n}\mathbf{t}_e)$$

Here D_e and S_e are determinants of the deviator and the shear tensor.

A similar representation holds for the stress tensor in the corresponding basis of the deviator plane.

2. We will assume that the material is ideally elastic (hyperelastic in another terminology) and its mechanical properties are determined by the elastic potential Φ (the specific strain energy). The differential of the elastic potential can be written as follows [3]:

$$d\Phi = K^*ede + 4G^*\bar{e}(\cos\omega^* d\bar{e} + \bar{e}\sin\omega^* d\varphi) \tag{2.1}$$

where K^* and G^* are the generalized moduli of bulk compression and shear, defined by the relations

$$\sigma = 3K^*e, \quad \bar{\sigma} = 2G^*\bar{e} \tag{2.2}$$

Differentiating the potential Φ using expression (2.1) with respect to the strain tensor, we obtain a non-linear law of elasticity for the given energy pair of tensors

$$\mathbf{\Sigma} = d\Phi / d\mathbf{E} = K^*e\mathbf{I} + 2G^*(\cos\omega^*\mathbf{D}_E + \sin\omega^*\mathbf{S}_E) \tag{2.3}$$

When differentiating the potential Φ we used the formulae

$$de / d\mathbf{E} = \mathbf{I}, \quad d\bar{e}^2 / d\mathbf{E} = \mathbf{D}_E, \quad 2\bar{e}^2 d\varphi / d\mathbf{E} = \mathbf{S}_E$$

Expression (2.3) is the expansion of the stress tensor in the strain tensor basis ($\mathbf{I}$, $\mathbf{D}_E$, $\mathbf{S}_E$). It follows from relations (1.1) and (2.3) that

$$\mathbf{D}_\Sigma = 2G^*(\cos\omega^*\mathbf{D}_E + \sin\omega^*\mathbf{S}_E), \quad \mathbf{S}_\Sigma = 2G^*(-\sin\omega^*\mathbf{D}_E + \cos\omega^*\mathbf{S}_E) \tag{2.4}$$

The law of elasticity (2.3) and relations (2.4) are easily inverted

$$\mathbf{E} = \frac{1}{9K^*}\sigma\mathbf{I} + \frac{1}{2G^*}(\cos\omega^*\mathbf{D}_\Sigma - \sin\omega^*\mathbf{S}_\Sigma) \tag{2.5}$$

$$\mathbf{D}_E = \frac{1}{2G^*}(\cos\omega^*\mathbf{D}_\Sigma - \sin\omega^*\mathbf{S}_\Sigma), \quad \mathbf{S}_E = \frac{1}{2G^*}(\sin\omega^*\mathbf{D}_\Sigma + \cos\omega^*\mathbf{S}_\Sigma) \tag{2.6}$$

Carrying out the operation of convolution, we obtain from (2.4) and (2.6)

$$\mathbf{D}_\Sigma : \mathbf{D}_E = \mathbf{S}_\Sigma : \mathbf{S}_E = 4G^*\bar{e}^2\cos\omega^*$$

$$\mathbf{D}_E : \mathbf{S}_\Sigma = -\mathbf{D}_\Sigma : \mathbf{S}_E = 4G^*\bar{e}^2\sin\omega^*$$

Hence we can determine the angle ω^*, in particular

$$\operatorname{tg}\omega^* = \frac{\mathbf{D}_E{:}\mathbf{S}_\Sigma}{\mathbf{D}_\Sigma{:}\mathbf{D}_E} = -\frac{\mathbf{D}_\Sigma{:}\mathbf{S}_E}{\mathbf{D}_\Sigma{:}\mathbf{D}_E}$$

To realize the model of a material with zero phase of the similitude of the stress and strain tensor deviators it is necessary and sufficient to satisfy the conditions: $\mathbf{D}_E : \mathbf{S}_\Sigma = -\mathbf{D}_\Sigma : \mathbf{S}_\Sigma = 0$, i.e. the convolution of the deviator and the shear tensor of the energy pair of tensors must equal zero.

The laws of elasticity (2.3) and (2.5) have the following form in the tensor components

$$\sigma_i = K^*e + 2G^*\left[\cos\omega^*\left(\varepsilon_i - \frac{1}{3}e\right) + \sin\omega^*\frac{1}{\sqrt{3}}(\varepsilon_k - \varepsilon_j)\right]$$

$$\varepsilon_i = \frac{1}{9K^*}\sigma + \frac{1}{2G^*}\left[\cos\omega^*\left(\sigma_i - \frac{1}{3}\sigma\right) - \sin\omega^*\frac{1}{\sqrt{3}}(\sigma_k - \sigma_j)\right]$$

Hence, the introduction of shear tensors has enabled us to write the non-linear law of elasticity in the simple and physically clear form (2.3) and (2.5). As a result, the linear relationship between the components of the stress and strain tensors on the principal axes has been made clear. The non-linearity is contained in the coefficients of the relations, which are functions of only the mixed Novozhilov invariants: the generalized moduli of elasticity and the similitude phase of the tensor deviators.

The law of elasticity was obtained by Novozhilov directly from the form of the functional relationship between two symmetrical coaxial tensors without resorting to a variation of the work. This law has the form [3]

$$\mathbf{\Sigma} = K^*e\mathbf{I} + 2G^*\left\{\cos\omega^*\mathbf{D}_E + \sin\omega^*\left[\operatorname{ctg}3\varphi\mathbf{D}_E - \frac{\sqrt{3}}{\bar{e}\sin3\varphi}\left(\mathbf{D}_E^2 - \frac{2}{3}\bar{e}^2\mathbf{I}\right)\right]\right\}$$

It can be shown, using (1.4), that the fairly complex expression in front of the square brackets in this formula can be converted into the shear tensor $\mathbf{S}_E$, i.e. it is a linear function of the strain in the frame of the principal axes, and its components have a real physical meaning. However, this proof has not previously been given.

If the system of coordinates is not smooth, the shear tensor can be determined in terms of the deviator using an expression which is equivalent to the last two equations of (1.3)

$$\mathbf{S}_E = \frac{D_e}{S_e}\mathbf{D}_E - \frac{2}{3}\bar{e}^2\frac{1}{S_e}\left(\mathbf{D}_E^2 - \frac{2}{3}\bar{e}^2\mathbf{I}\right)$$

To obtain the angles of the form of the tensor and the similitude phase of the deviators one can use the formulae

$$\operatorname{tg}3\varphi = \frac{S_e}{D_e}, \quad \operatorname{tg}3\psi = \frac{S_\sigma}{D_\sigma}, \quad \operatorname{tg}3\omega = \frac{D_e S_\sigma - D_\sigma S_e}{D_e D_\sigma + S_e S_\sigma} \tag{2.7}$$

where the determinants of the deviators and the shear tensors are related by the expressions

$$D_e^2 + S_e^2 = \frac{4}{27}\bar{e}^6, \quad D_\sigma^2 + S_\sigma^2 = \frac{4}{27}\bar{\sigma}^6 \tag{2.8}$$

Relations (2.7) and (2.8) were derived using (1.4). It follows from (2.7) that the angles of the form of the tensors and their difference (the similitude phase of the deviators) lie in the range $(-\pi/6, +\pi/6)$. It was previously assumed [3] that the angle ω^* varies in the range $(-\pi/3, +\pi/3)$. Formulae (2.7) are convenient in that they are invariant. To determine the angles of the form of the tensors from (2.7) one cannot, as is usually done, arrange the stresses and strains in order of their magnitude (introduce relations of the form $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$).

3. Linear relations between the tensors considered arise not only in the frame of the principal strain axes. Suppose that, in an elastic body, we introduce a system of orthogonal curvilinear coordinates with a vector basis $\mathbf{e}_i$ ($i = 1, 2, 3$), where only one direction $\mathbf{e}_3$ is the principal direction. This situation arises in plane or axi-symmetrical strain, with a plane stressed state. We will show that in this case the relationship between the stress and the strain is linear (the non-linearity is contained in the coefficients, which are functions of the invariants). From relations (1.3) we obtain the shear tensor (ε_{ij} are the components of the strain tensor $\mathbf{E}$)

$$\begin{aligned}\mathbf{S}_E = &-\frac{1}{2}\left[s + \frac{d}{s}(\varepsilon_{11} - \varepsilon_{22})\right]\mathbf{e}_1\mathbf{e}_1 - \frac{d}{s}\varepsilon_{12}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) - \\ &-\frac{1}{2}\left[s - \frac{d}{s}(\varepsilon_{11} - \varepsilon_{22})\right]\mathbf{e}_2\mathbf{e}_2 + s\mathbf{e}_3\mathbf{e}_3\end{aligned} \tag{3.1}$$

where $d = \varepsilon_{33} - e/3$ and $\sqrt{(4e^{-2}/3 - d^2)}$ are the components of the deviator and of the shear tensor in the principal direction. These components are invariants, so that the shear tensor $\mathbf{S}_\mathrm{E}$ and the relations of the law of elasticity (2.3) are linear in the strains.

4. We will obtain the non-linear law of elasticity for the tensors of the true stresses and strains. We can derive the energy pair considered by converting the variation of the work of deformation

$$\delta W = \mathbf{\Sigma} : \delta\mathbf{E} = (\mathbf{G}^{-1}J\mathbf{T}\mathbf{G}) : \delta\mathbf{R}$$

where $\mathbf{G}$ is the gradient of the deformation, $J = \det \mathbf{G}$ is the multiplicity of the change in volume, $\mathbf{T}$ is the tensor of the true stresses, $\mathbf{R} = \ln \mathbf{\Lambda}$ is the tensor of the true strains, and $\mathbf{\Lambda}$ is the tensor of the elongation multiplicities, where $\mathbf{\Lambda}^2 = \mathbf{I} + 2\mathbf{E}$. The tensor $\mathbf{T}^* = \mathbf{G}^{-1}J\mathbf{T}\mathbf{G} = \mathbf{Q}^T J\mathbf{T}\mathbf{Q}$, comprising the energy pair of tensor $\mathbf{R}$, differs from the tensor $\mathbf{T}$ by the presence of rotation of the basis axes. This is due to the fact that the tensor $\mathbf{T}$ refers to the actual configuration, while the tensor $\mathbf{T}^*$ refers to the initial configuration. Rotation is carried out using the orthogonal tensor $\mathbf{Q}$, which participates in the polar expansion $\mathbf{G} = \mathbf{Q}\mathbf{\Lambda}$.

The differential of the elastic potential for the true stresses and strains has the form [2]

$$d\Phi = KJrdr + 4GJ\bar{r}(\cos\omega d\bar{r} + \sin\omega\bar{r}d\varphi_R) \tag{4.1}$$

where $(r, \bar{r}, \varphi_R)$ are the invariants of the tensor **R**, analogous to the invariants $(e, \bar{e}, \varphi)$ of the tensor **E**, in particular $r = \ln J$, K and G are the generalized moduli of elasticity and $\omega = \psi_T - \varphi_R$ is the difference in the angles of the form of the tensors (the similitude phase of the deviator). From (4.1) we obtain the law of elasticity

$$\mathbf{T}^* = d\Phi / d\mathbf{R} = KJr\mathbf{I} + 2GJ(\cos\omega\mathbf{D}_R + \sin\omega\mathbf{S}_R) \tag{4.2}$$

$$t = 3Kr, \quad \bar{t} = 2G\bar{r}$$

where t and $\bar{t}$ are the invariants of the tensor **T**, similar to the invariants σ and $\bar{\sigma}$ of the tensor $\boldsymbol{\Sigma}$.

For the deviators $\mathbf{D}_T$ and $\mathbf{D}_R$ and the shear tensors $\mathbf{S}_T$ and $\mathbf{S}_R$ all the formulae which were obtained above for the deviators and the shear tensors of the energy pair $(\boldsymbol{\Sigma}, \mathbf{E})$ hold (only the invariants need to be replaced).

In particular

$$\mathbf{D}_T = 2GJ(\cos\omega\mathbf{D}_R + \sin\omega\mathbf{S}_R), \quad \mathbf{S}_T = 2GJ(-\sin\omega\mathbf{D}_R + \cos\omega\mathbf{S}_R) \tag{4.3}$$

In the components of the tensors the law of elasticity (4.2) has the form

$$t_i = Kr + 2G\left[\cos\omega\left(r_i - \frac{1}{3}r\right) + \sin\omega\frac{1}{\sqrt{3}}(r_k - r_j)\right] \tag{4.4}$$

Here t_i and r_i are the components of the tensors of the true stresses and strains in the frame of the principal axes.

It is easy to write relations that are the inverse of (4.2)–(4.4).

When only one of the coordinate directions is the principal direction, the linear dependence of the true stresses and strains is established by a method similar to that considered above. The shear tensor for the true strains will be analogous to tensor (3.1).

5. Three energy pairs of tensors and their corresponding laws of elasticity were considered in [2] and estimates were given of their advantages and disadvantages for practical applications. The first and second versions of the relations were given above—formulae (2.3) and (4.2). The third version relates to the energy pair $(\mathbf{B}, \boldsymbol{\Lambda})$, where $\mathbf{B} = \mathbf{G}^{-1}J\mathbf{TQ}$ is the Biot stress tensor and $\boldsymbol{\Lambda}$ of the elongation multiplicities.

We share the opinion of Novozhilov [2] with regard to the first two and the third versions of the constitutive equations. The first version is convenient from the mathematical point of view, since Green's strain tensor is described fairly simply in terms of the displacement. However, the generalized moduli of elasticity, which occur in (2.3), lose their physical meaning for large deformations. The second version is physically clear, since the quantities K and G in (4.2) retain their physical meaning of the bulk compression modulus and the shear modulus for large deformations. But this version is less convenient mathematically, since the tensor of the true strains is complex. Hence, the first two versions of the governing equations, in effect, supplement one another and justify us in thinking that they can both find application [2]. As regards the equations for the Biot tensors and the elongation multiplicities, Novozhilov writes: "the third version is generally not rational since it possesses all the drawbacks and none of the advantages of the two previous versions".

The law of elasticity in Novozhilov's form has been used [4] to construct a non-linear theory of an elastomer layer. The two-dimensional equations of the layer were obtained by the asymptotic method. Certain kinematic hypotheses, corresponding to the deformation conditions of these layers in multilayer rubber-metal structures, were also assumed. The similitude phase of the tensor deviators in the elasticity law was assumed to be zero. The second-degree value of the similitude phase in certain equations, compared with the generalized moduli of elasticity, was pointed out earlier [1].

We will consider one example of the practical application of the results obtained in constructing approximate relations between the stresses and strains. Using the first and second versions of the constitutive equations (2.3) and (4.2) we can derive the law of elasticity, which combines their advantages and is free from the disadvantages mentioned above. The defect of formulae (2.3) is the fact that the generalized moduli lose their physical meaning at large strains. If we express the moduli K^* and G^* in terms of the moduli K and G, which occur in (4.2) and which retain their physical meaning at large strains, the drawbacks will be eliminated. This operation can be carried out using the formula $Jt_i = \lambda_i^2\sigma_i$, which relates the true strains and the Piola–Kirchhoff strains. By calculating the first and second invariants of the stress tensors from (2.3) and (4.2), we can establish the required relation between the generalized moduli. The results are given below in the approximate version as it applies to elastomer materials.

For rubber-like materials the bulk compression modulus is three-four orders of magnitude greater than the shear modulus (Poisson's ratio is close to 0.5). Omitting small terms of the order of the ratio of the moduli in the relations between the generalized moduli of elasticity, we obtain

$$KJ\ln J = K^*e\left(1+\frac{2}{3}e\right)+\frac{8}{3}G^*\bar{e}^2$$

$$GJ\bar{r} = \left[K^*e+\left(1+\frac{2}{3}e+\frac{2}{\sqrt{3}}\bar{e}\right)G^*\right]\bar{e}$$

If we assume that the generalized moduli K and G are known functions of the invariants of the true strain tensor, the generalized moduli K^* and G^* also become known.

Using relation (4) we obtain the first invariant of the tensor

$$p=\frac{1}{3}(t_1+t_2+t_3)=K\ln J$$

We know that the law of compressibility $p = K\ln J$, for constant modulus K, is applicable up to pressures of the order of 40–50 MPa. Hence, K can be assumed to be constant over this pressure range. If the similitude phase of the deviators is small (for example, for a Hencky material, used in the theory of plasticity), the expression GJ will be a function only of the invariant $\bar{r}$—the intensity of the shear strain, which can be found by simple experiments.

6. The relations obtained for the energy pairs of tensors have a common mathematical form. They are satisfied for any pair of symmetrical coaxial tensors. In fact, one can change to the theory of elasticity when the dependence of the moduli of elasticity on the tensor invariants is specified or obtained experimentally. The linear dependence of the stresses and strains is obtained in two versions: when the system of coordinates is the principal one and when there is only one principal direction.

We will indicate some areas of possible application of the results: in a number of theoretical investigations and, primarily, when constructing elastic potentials or non-linear laws of elasticity, when solving boundary-value problems of the non-linear theory of elasticity, particularly those where it is possible to use the principal coordinates (the class of such problems is fairly wide, and some of them have been indicated earlier), when processing experimental data, in particular, to obtain the dependence of the generalized moduli of elasticity on the invariants [2], and when setting up models of small dimension of the non-linear theory of elasticity for a layer, plates, shells, membranes and rods.

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